Metric fluctuations of an evaporating black hole from back reaction of stress tensor fluctuations

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This paper delineates the first steps in a systematic quantitative study of the spacetime fluctuations induced by quantum fields in an evaporating black hole under the stochastic gravity program. The central object of interest is the noise kernel, which is the symmetrized two-point quantum correlation function of the stress tensor operator. As a concrete example we apply it to the study of the spherically-symmetric sector of metric perturbations around an evaporating black hole background geometry. For macroscopic black holes we find that those fluctuations grow and eventually become important when considering sufficiently long periods of time (of the order of the evaporation time), but well before the Planckian regime is reached. In addition, the assumption of a simple correlation between the fluctuations of the energy flux crossing the horizon and far from it, which was made in earlier work on spherically-symmetric induced fluctuations, is carefully scrutinized and found to be invalid. Our analysis suggests the existence of an infinite amplitude for the fluctuations when trying to localize the horizon as a three-dimensional hypersurface, as in the classical case, and, as a consequence, a more accurate picture of the horizon as possessing a finite effective width due to quantum fluctuations. This is supported by a systematic analysis of the noise kernel in curved spacetime smeared with different functions under different conditions, the details are collected in the appendices. This case study shows a pathway for probing quantum metric fluctuations near the horizon and understanding their physical meaning.

I. INTRODUCTION

Studying the dynamics of quantum fields in a fixed curved spacetime, Hawking found that black holes emit thermal radiation with a temperature inversely proportional to their mass [1]. When the back reaction of the quantum fields on the spacetime dynamics is included, one expects that the mass of the black hole decreases as thermal radiation at higher and higher temperatures is emitted. This picture, which constitutes the process known as black hole evaporation, is indeed obtained from semiclassical gravity calculations which are believed to be valid at least before the Planckian scale is reached [2, 3].

Semiclassical gravity [4, 5, 6] is a mean field description that neglects the fluctuations of the spacetime geometry. However, a number of studies have suggested the existence of large fluctuations near black hole horizons [7, 8, 9, 10] (and even instabilities [11]) with characteristic time-scales much shorter than the black hole evaporation time. In all of them[1] either states which are singular on the horizon (such as the Boulware vacuum for Schwarzschild spacetime) were explicitly considered, or fluctuations were computed with respect to those states and found to be large near the horizon. Whether these huge fluctuations are of a generic nature or an artifact arising from the consideration of states singular on the horizon is an issue that deserves further investigation. By contrast, the fluctuations for states regular on the horizon were estimated in Ref. [12] and found to be small even when integrated over a time of the order of the evaporation time.

These apparently contradictory claims and the fact that most claims on black hole horizon fluctuations were based on qualitative arguments and/or semi-quantitative estimates prompted us here to strive for a more quantitative and self-consistent description[2]. For this endeavor we follow the stochastic gravity program [15, 16, 17, 18, 19]. We will consider the fluctuations of metric perturbations around a black hole geometry interacting with a quantum scalar field whose stress tensor drives the dynamics. The evolution of the mean background geometry is given by the semiclassical Einstein equation (with self-consistent back reaction from the expectation value of the stress tensor) while the metric

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[1] At least those which provide a relativistic description. The argument in Refs. [7, 8] is based on a non-relativistic description and it is not obvious how to make some of our statements precise in that case. However, a natural generalization to the relativistic case is provided in Ref. [10], which does fall into this category.

[2] Previous attempts on this problem with similar emphasis by Raval, Sinha and one of us have appeared in Refs. [13, 14]. The apparent difference between the conclusions in Ref. [13] and what is reported here will be explained below.

fluctuations obey an Einstein-Langevin equation [20, 21, 22, 23] with a Gaussian stochastic source whose correlation function is given by the noise kernel, which characterizes the fluctuations of the stress tensor of the quantum fields. In contrast to the claims made before, we find here that even for states regular on the horizon the accumulated fluctuations become significant by the time the black hole mass has changed substantially, but well before reaching the Planckian regime. Our result is different from those obtained in prior studies, but in agreement with earlier work by Bekenstein [24].

The stochastic gravity program provides perhaps the best available framework to study quantum metric fluctuations, because while semiclassical gravity is a mean field description that does not take into account quantum metric fluctuations, the Einstein-Langevin equation enables one to solve for the dynamics of metric fluctuations induced by the fluctuations of the stress tensor of the quantum fields. Furthermore, the correlation functions that one obtains are equivalent to the quantum correlation functions for the metric perturbations around the semiclassical background that would follow from a quantum field theory treatment, up to a given order in an expansion in terms of the inverse number of fields [25, 26]. The quantization of these metric perturbations should be understood in the framework of a low-energy effective field theory approach to quantum gravity [27], which is expected to provide reliable results for phenomena involving typical length-scales much larger than the Planck length even if the microscopic details of the theory at Planckian scales are not known.[3]

A crucial relation assumed in previous investigations [12, 24][4] of the problem of metric fluctuations driven by quantum matter field fluctuations of states regular on the horizon (as far as the expectation value of the stress tensor is concerned) is the existence of correlations between the outgoing energy flux far from the horizon and a negative energy flux crossing the horizon, based on energy conservation arguments. Using semiclassical gravity, such correlations have been confirmed for the expectation value of the energy fluxes, provided that the mass of the black hole is much larger than the Planck mass. However, a more careful analysis, summarized in Sec. IV, shows that no such simple connection exists for energy flux fluctuations. It also reveals that the fluctuations on the horizon are in fact divergent unless it is treated as an object with a finite width rather than a three-dimensional hypersurface, as in the classical case, and one needs to find an appropriate way of probing the metric fluctuations near the horizon and extracting physically meaningful information. This is a new challenge in the study of metric fluctuations which demands some careful thoughts on what they mean physically and how they can be probed operationally. In Appendices B and C we give a systematic analysis of the noise kernel in curved spacetime smeared with different functions under different conditions. The non-existence of this commonly invoked relation in this whole subject matter illustrates the limitations of heuristic arguments and the necessity of a detailed and consistent formalism to study the fluctuations near the horizon, in terms of their magnitude, how they are measured and their consequences.

A few technical remarks are in order to delimit the problem under study: First, we will restrict our attention to the spherically-symmetric sector of metric fluctuations, which necessarily implies a partial description of the fluctuations. That is because, contrary to the case for semiclassical gravity solutions, even if one starts with spherically-symmetric initial conditions, the stress tensor fluctuations will induce fluctuations involving higher multipoles. Thus, the multipole structure of the fluctuations is far richer than that of spherically-symmetric semiclassical gravity solutions, but this also means that obtaining a complete solution (including all multipoles) for fluctuations rather than the mean value is much more difficult.

Second, for black hole masses much larger than the Planck mass (otherwise the effective field theory description will break down anyway), one can introduce a useful adiabatic approximation involving inverse powers of the black hole mass. To obtain results to lowest order, it is sufficient to compute the expectation value of the stress tensor operator and its correlation functions in Schwarzschild spacetime. The corrections, proportional to higher powers of the inverse mass, can be neglected for sufficiently massive black holes.

Third, when studying the dynamics of induced metric fluctuations, the additional contribution to the stress tensor expectation value which results from evaluating it using the perturbed metric is often neglected. In the consideration of fluctuations for an evaporating black hole such a term (which will be denoted by $\langle \hat{T}_{ab}^{(1)}[g+h] \rangle_{\mathrm{ren}}$ in Sec. III) becomes important when it builds up for long times. The importance of this term is clear when comparing with the simple estimate made by Wu and Ford in Ref. [12], where $\langle \hat{T}_{ab}^{(1)}[g+h] \rangle_{\mathrm{ren}}$ was neglected and the fluctuations were found to be small even when integrated over long times, of the order of the evaporation time of the black hole.

The paper is organized as follows. In Sec. II we briefly review the results for the evolution of the mean field geometry of an evaporating black hole obtained in the context of semiclassical gravity. The framework of stochastic gravity

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[3] This approach has been mainly applied to weak field situations, such as the study of quantum corrections to the Newtonian potential for particles in a Minkowski background [28, 29]. However, it is particularly interesting to apply it also to strong field situations involving cosmological [30] or black hole spacetimes.

[4] See, however, Refs. [31, 32], where those correlators were shown to vanish in an effectively two-dimensional model.

is then applied in Sec. III to the study of the spherically-symmetric sector of fluctuations around the semiclassical gravity solution for an evaporating black hole. It has been previously assumed that an exact correlation between the fluctuations of the negative energy flux crossing the horizon and the flux far from it exists. In this paper we want to question this assumption, but in the presentation in Sec. III we accept temporarily such a working hypothesis just so that we can have the common ground to compare our results with those in the literature. In Sec. IV, we present a careful analysis of this assumption, and show that this supposition is invalid. Further details of this proof can be found in the Appendices B and C . Finally, in Sec. V we discuss several implications of our results and suggest some directions for further investigation.

Throughout the paper we use Planckian units with $\hbar = c = G = 1$ and the $(+, +, +)$ convention of Ref. [33]. We also make use of the abstract index notation of Ref. [34]. Latin indices denote abstract indices, whereas Greek indices are employed whenever a particular coordinate system is considered.

II.   MEAN EVOLUTION OF AN EVAPORATING BLACK HOLE

Semiclassical gravity provides a mean field description of the dynamics of a classical spacetime where the gravitational back reaction of quantum matter fields is included self-consistently [4, 5, 6]. It is believed to be applicable to situations involving length-scales much larger than the Planck scale and for which the quantum back-reaction effects due to the metric itself can be neglected as compared to those due to the matter fields. The dynamics of the metric g_{ab} is governed by the semiclassical Einstein equation:

$$G_{ab}\left[g\right] = \kappa \left\langle \hat{T}_{ab}[g] \right\rangle_{\mathrm{ren}}, \tag{1}$$

where $\langle \hat{T}_{ab}[g] \rangle_{\mathrm{ren}}$ is the renormalized expectation value of the stress tensor operator of the quantum matter fields and $\kappa = 8\pi/m_{\mathrm{p}}^2$ with m_{p}^2 being the Planck mass. One must solve both the semiclassical Einstein equation and the equation of motion for the matter fields evolving in that geometry, whose solution is needed to evaluate $\langle \hat{T}_{ab}[g] \rangle_{\mathrm{ren}}$ self-consistently.

An important application of semiclassical gravity is the study of black hole evaporation due to the back reaction of the Hawking radiation emitted by the black hole on the spacetime geometry. This has been studied in some detail for spherically symmetric black holes [2, 3]. For a general spherically-symmetric metric there always exists a system of coordinates in which it takes the form

$$ds^2 = -e^{2\psi(v,r)}(1 - 2m(v,r)/r)dv^2 + 2e^{\psi(v,r)}dvdr + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{2}$$

This completely fixes the gauge freedom under local diffeomorphism transformations except for an arbitrary function of v that can be added to the function $\psi(v,r)$ and is related to the freedom in reparametrizing v (we will see below how this can also be fixed). In general this metric exhibits an *apparent horizon*, where the expansion of the outgoing radial null geodesics vanishes and which separates regions with positive and negative expansion for those geodesics, at those radii that correspond to (odd degree) zeroes of the vv metric component. We denote the location of the apparent horizon by $r_{\mathrm{AH}}(v) = 2M(v)$, where $M(v)$ satisfies the equation $2m(2M(v), v) = 2M(v)$.

Spherical symmetry implies that the components $T_{\theta r}$, $T_{\theta v}$, $T_{\varphi r}$ and $T_{\varphi v}$ vanish and the remaining components are independent of the angular coordinates. Under these conditions the various components of the semiclassical Einstein equation associated with the metric in Eq. (2) become

$$\frac{\partial m}{\partial v} = 4\pi r^2 T_v^r, \tag{3}$$

$$\frac{\partial m}{\partial r} = -4\pi r^2 T_v^v, \tag{4}$$

$$\frac{\partial \psi}{\partial r} = 4\pi r T_{rr}, \tag{5}$$

where in the above and henceforth we simply use $T_{\mu\nu}$ to denote the expectation value $\langle \hat{T}_{\mu\nu}[g] \rangle_{\mathrm{ren}}$ and employ Planckian units (with $m_{\mathrm{p}}^2 = 1$). Note that the arbitrariness in ψ can be eliminated by choosing a parametrization of v such that ψ takes a particular value at a given radius (we will choose that it vanishes at $r = 2M(v)$, where the apparent horizon is located); ψ is then entirely fixed by Eq. (5).

Solving Eqs. (3)-(5) is no easy task. However, one can introduce a useful adiabatic approximation in the regime where the mass of the black hole is much larger than the Planck mass, which is in any case a necessary condition for the semiclassical treatment to be valid. What this entails is that when $M \gg 1$ (remember that we are using

Planckian units) for each value of v one can simply substitute $T_{\mu\nu}$ by its "parametric value" – by this we mean the expectation value of the stress energy tensor of the quantum field in a Schwarzschild black hole with a mass corresponding to $M(v)$ evaluated at that value of v. This is in contrast to its dynamical value, which should be determined by solving self-consistently the semiclassical Einstein equation for the spacetime metric and the equations of motion for the quantum matter fields. This kind of approximation introduces errors of higher order in $L_{\mathrm{H}} \equiv B/M^2$ (B is a dimensionless parameter that depends on the number of massless fields and their spins and accounts for their corresponding grey-body factors; it has been estimated to be of order 10^{-4} [35]), which are very small for black holes well above Planckian scales. These errors are due to the fact that $M(v)$ is not constant and that, even for a constant $M(v)$, the resulting static geometry is not exactly Schwarzschild because the vacuum polarization of the quantum fields gives rise to a non-vanishing $\langle \hat{T}_{ab}[g]\rangle_{\mathrm{ren}}$ [36].

The expectation value of the stress tensor for Schwarzschild spacetime has been found to correspond to a thermal flux of radiation (with $T_v^r = L_{\mathrm{H}}/(4\pi r^2)$) for large radii and of order L_{H} near the horizon[5] [37, 38, 39, 40, 41]. This shows the consistency of the adiabatic approximation for $L_{\mathrm{H}} \ll 1$: the right-hand side of Eqs. (3)-(5) contains terms of order L_{H} and higher, so that the derivatives of $m(v,r)$ and $\psi(v,r)$ are indeed small. Furthermore, one can use the v component of the stress-energy conservation equation

$$\frac{\partial\left(r^2 T_v^r\right)}{\partial r} + r^2 \frac{\partial T_v^v}{\partial v} = 0, \tag{6}$$

to relate the T_v^r components on the horizon and far from it. Integrating Eq. (6) radially, one gets

$$(r^2 T_v^r)(r = 2M(v), v) = (r^2 T_v^r)(r \approx 6M(v), v) + O(L_{\mathrm{H}}^2), \tag{7}$$

where we considered a radius sufficiently far from the horizon, but not arbitrarily far (i.e. $2M(v) \ll r \ll M(v)/L_{\mathrm{H}}$). The second condition is necessary to ensure that the size of the horizon has not changed much since the value of v' at which the radiation crossing the sphere of radius r at time v left the region close to the horizon. Note that while in the nearly flat region (for large radii) T_v^r corresponds to minus the outgoing energy flux crossing the sphere of radius r, on the horizon, where $ds^2 = 2e^{\psi(v,r)}dvdr + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$, T_v^r equals T_{vv}, which corresponds to the null energy flux crossing the horizon. Hence, Eq. (7) relates the positive energy flux radiated away far from the horizon and the negative energy flux crossing the horizon. Taking into account this connection between energy fluxes and evaluating Eq. (3) on the apparent horizon, we finally get the equation governing the evolution of its size:

$$\frac{dM}{dv} = -\frac{B}{M^2}. \tag{8}$$

Unless $M(v)$ is constant, the event horizon and the apparent horizon do not coincide. However, in the adiabatic regime their radii are related, differing by a quantity of higher order in L_{H}: $r_{\mathrm{EH}}(v) = r_{\mathrm{AH}}(v)\left(1 + O(L_{\mathrm{H}})\right)$.

We close this section with an explanation of why we did not have to deal with terms involving higher-order derivatives and even non-local terms when considering the expectation value of the stress tensor as one would expect for geometries with sufficiently arbitrary spacetime dependence of certain metric components. (This can be seen in explicit calculations for arbitrary Robertson-Walker geometries [15, 42] or arbitrary small metric perturbations around specific backgrounds [43, 44].) In our case such non-local and higher-order derivative terms would also appear in the exact expression of the stress tensor expectation value for arbitrary $m(v,r)$ and $\psi(v,r)$ functions. However, the adiabatic approximation for $M \gg 1$ that we have employed effectively gets rid of them since one can replace the higher-derivative terms using Eqs. (3)-(5) recursively and taking into account that the terms on the right-hand side are of order $1/M^2$. Therefore, higher-order derivative terms correspond to higher powers of $1/M^2$ and are highly suppressed for $M \gg 1$. Note that this argument, which is based on the black hole size being much larger than the Planck length, is closely related to the order reduction prescription [6, 45] that is often used to deal with higher-derivative terms in semiclassical gravity and other back-reaction problems.

III. SPHERICALLY-SYMMETRIC INDUCED FLUCTUATIONS

There are situations in which the fluctuations of the stress tensor operator and the metric fluctuations that they induce may be important, so that the mean field description provided by semiclassical gravity is incomplete and even

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[5] The natural quantum state for a black hole formed by gravitational collapse is the Unruh vacuum, which corresponds to the absence of incoming radiation far from the horizon. The expectation value of the stress tensor operator for that state is finite on the future horizon of Schwarzschild, which is the relevant one when identifying a region of the Schwarzschild geometry with the spacetime outside the collapsing matter for a black hole formed by gravitational collapse.

fails to capture the most relevant phenomena (the generation of primordial cosmological perturbations constitutes a clear example of that). *Stochastic gravity* [15, 16, 17, 18, 19] provides a framework to study those fluctuations. Its centerpiece is the Einstein-Langevin equation [20, 21, 22, 23]

$$G_{ab}^{(1)}\left[g+h\right] = \kappa \left\langle \hat{T}_{ab}^{(1)}[g+h]\right\rangle_{\mathrm{ren}} + \kappa\,\xi_{ab}\left[g\right],\tag{9}$$

which governs the dynamics of the metric fluctuations around a background metric g_{ab} that corresponds to a given solution of semiclassical gravity. The superindex (1) indicates that only the terms linear in the metric perturbations should be considered, and ξ_{ab} is a Gaussian stochastic source with vanishing expectation value and correlation function[6] $\langle\xi_{ab}(x)\xi_{cd}(x')\rangle_\xi = (1/2)\langle\{\hat{t}_{ab}(x),\hat{t}_{cd}(x')\}\rangle$ (with $\hat{t}_{ab} \equiv \hat{T}_{ab} - \langle\hat{T}_{ab}\rangle$), where the term on the right-hand side, which accounts for the stress tensor fluctuations within this Gaussian approximation, is commonly known as the noise kernel and denoted by $N_{abcd}(x, x')$. In this framework the metric perturbations are still classical but stochastic. Nevertheless, one can show that the correlation functions for the metric perturbations that one obtains in stochastic gravity are equivalent through order $1/N$ to the quantum correlation functions that would follow from a quantum field theory treatment when considering a large number of fields N [25, 26]. In particular, the symmetrized two-point function consists of two contributions: *intrinsic* and *induced* fluctuations. The intrinsic fluctuations are a consequence of the quantum width of the initial state of the metric perturbations, and they are obtained in stochastic gravity by averaging over the initial conditions for the solutions of the homogeneous part of Eq. (9) distributed according to the reduced Wigner function associated with the initial quantum state of the metric perturbations. On the other hand, the induced fluctuations are due to the quantum fluctuations of the matter fields interacting with the metric perturbations, and they are obtained by solving the Einstein-Langevin equation using a retarded propagator with vanishing initial conditions.

In this section we study the spherically-symmetric sector [*i.e.*, the monopole contribution, which corresponds to $l = 0$, in a multipole expansion in terms of spherical harmonics $Y_{lm}(\theta, \phi)$] of metric fluctuations for an evaporating black hole. In this case only induced fluctuations are possible. The fact that intrinsic fluctuations cannot exist can be clearly seen if one neglects vacuum polarization effects, since Birkhoff's theorem forbids the existence of spherically-symmetric free metric perturbations in the exterior vacuum region of a spherically-symmetric black hole that keep the ADM mass constant. Even when vacuum polarization effects are included, spherically-symmetric perturbations, characterized by $m(v, r)$ and $\psi(v, r)$, are not independent degrees of freedom. This follows from Eqs. (3)-(5), which can be regarded as constraint equations.

The fluctuations of the stress tensor are inhomogeneous and non-spherically-symmetric even if we choose a spherically-symmetric vacuum state for the matter fields (spherical symmetry simply implies that the angular dependence of the noise kernel in spherical coordinates is entirely given by the relative angle between the spacetime points x and x'). This means that, in contrast to the semiclassical gravity case, projecting onto the $l = 0$ sector of metric perturbations does not give an exact solution of the Einstein-Langevin equation in the stochastic gravity approach that we have adopted here. Nevertheless, restricting to spherical symmetry in this way gives more accurate results than two-dimensional dilaton-gravity models resulting from simple dimensional reduction [46, 47, 48]. This is because we project the solutions of the Einstein-Langevin equation just at the end, rather than considering only the contribution of the s-wave modes to the classical action for both the metric and the matter fields from the very beginning. Hence, an infinite number of modes for the matter fields with $l \neq 0$ contribute to the $l = 0$ projection of the noise kernel, whereas only the s-wave modes for each matter field would contribute to the noise kernel if dimensional reduction had been imposed right from the start, as done in Refs. [31, 32, 49] as well as in studies of two-dimensional dilaton-gravity models.

The Einstein-Langevin equation for the spherically-symmetric sector of metric perturbations can be obtained by considering linear perturbations of $m(v, r)$ and $\psi(v, r)$, projecting the stochastic source that accounts for the stress tensor fluctuations to the $l = 0$ sector, and adding it to the right-hand side of Eqs. (3)-(5). We will focus our attention on the equation for the evolution of $\eta(v, r)$, the perturbation of $m(v, r)$:

$$\frac{\partial(m+\eta)}{\partial v} = -\frac{B}{(m+\eta)^2} + 4\pi r^2\xi_v^r + O\left(L_{\mathrm{H}}^2\right),\tag{10}$$

which reduces, after neglecting terms of order L_{H}^2 or higher, to the following equation to linear order in η:

$$\frac{\partial\eta}{\partial v} = \frac{2B}{m^3}\eta + 4\pi r^2\xi_v^r.\tag{11}$$

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[6] Throughout the paper we use the notation $\langle\ldots\rangle_\xi$ for stochastic averages over all possible realizations of the source ξ_{ab} to distinguish them from quantum averages, which are denoted by $\langle\ldots\rangle$.

It is important to emphasize that in Eq. (10) we assumed that the change in time of $\eta(v, r)$ is sufficiently slow so that the adiabatic approximation employed in the previous section to obtain the mean evolution of $m(v, r)$ can also be applied to the perturbed quantity $m(v, r) + \eta(v, r)$. This is guaranteed as long as the term corresponding to the stochastic source is of order L_{H} or higher, a point that will be discussed below.

Obtaining the noise kernel which determines the correlation function for the stochastic source is highly nontrivial even if we compute it on the Schwarzschild spacetime, which is justified in the adiabatic regime for the background geometry. As implicitly done in prior work (for instance in Refs. [12, 24]; see, however, Refs. [31, 32]), we will assume in this section that the fluctuations of the radiated energy flux far from the horizon are exactly correlated with the fluctuations of the negative energy flux crossing the horizon. This is a crucial assumption which implies an enormous simplification and allows a direct comparison with the results in the existing literature, and its validity will be analyzed more carefully in the next section.[7]

Since the generation of Hawking radiation is especially sensitive to what happens near the horizon, from now on we will concentrate on the metric perturbations near the horizon[8] and consider $\eta(v) = \eta(v, 2M(v))$. Assuming that the fluctuations of the energy flux crossing the horizon and those far from it are exactly correlated, from Eq. (11) we have

$$\frac{d\eta(v)}{dv} = \frac{2B}{M^3(v)}\eta(v) + \xi(v),\tag{12}$$

where $\xi(v) \equiv (4\pi r^2 \xi_v^r)(v, r \approx 6M(v))$. The correlation function for the spherically-symmetric fluctuation $\xi(v)$ is determined by the integral over the whole solid angle of the $N^r{}_v{}^r{}_v$ component of the noise kernel, which is given by $(1/2)\langle\{\hat{t}_v^r(x), \hat{t}_v^r(x')\}\rangle$. The $l = 0$ fluctuations of the energy flux of Hawking radiation far from a black hole formed by gravitational collapse, characterized also by $(1/2)\langle\{\hat{t}_v^r(x), \hat{t}_v^r(x')\}\rangle$ averaged over the whole solid angle, have been studied in Ref. [12]. Its main features are a correlation time of order M and a characteristic fluctuation amplitude of order ϵ_0/M^4 (this is the result of smearing the stress tensor two-point function, which diverges in the coincidence limit, over a period of time of the order of the correlation time). The order of magnitude of ϵ_0 has been estimated to lie between $0.1B$ and B [12, 24]. For simplicity, we will consider quantities smeared over a time of order M. We can then introduce the Markovian approximation $(\epsilon_0/M^3(v))\delta(v - v')$, which coarse-grains the information on features corresponding to time-scales shorter than the correlation time M. Under those conditions $r^2 \xi_v^r$ is of order $1/M^2$ and thus the adiabatic approximation made when deriving Eq. (10) is justified.

The stochastic equation (12) can be solved in the usual way and the correlation function for $\eta(v)$ can then be computed. Alternatively, one can follow Bekenstein [24] and derive directly an equation for $\langle\eta^2(v)\rangle_\xi$. This is easily done multiplying Eq. (12) by $\eta(v)$ and taking the expectation value. The result is

$$\frac{d}{dv}\langle\eta^2(v)\rangle_\xi = \frac{4B}{M^3(v)}\langle\eta^2(v)\rangle_\xi + 2\langle\eta(v)\xi(v)\rangle_\xi.\tag{13}$$

For delta-correlated noise (the Stratonovich prescription is the appropriate one in this case), $\langle\eta(v)\xi(v)\rangle_\xi$ equals one half the time-dependent coefficient multiplying the delta function $\delta(v - v')$ in the expression for $\langle\xi(v)\xi(v')\rangle_\xi$, which is given by $\epsilon_0/M^3(v)$ in our case. Taking that into account, Eq. (13) becomes

$$\frac{d}{dv}\langle\eta^2(v)\rangle_\xi = \frac{4B}{M^3(v)}\langle\eta^2(v)\rangle_\xi + \frac{\epsilon_0}{M^3(v)}.\tag{14}$$

Finally, it is convenient to change from the v coordinate to the mass function $M(v)$ for the background solution. Eq. (14) can then be rewritten as

$$\frac{d}{dM}\langle\eta^2(M)\rangle_\xi = -\frac{4}{M}\langle\eta^2(M)\rangle_\xi - \frac{(\epsilon_0/B)}{M}.\tag{15}$$

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[7] This simple relation between the energy flux crossing the horizon and the flux far from it is valid for the expectation value of the stress tensor, which is based on an energy conservation argument for the T_v^r component. In most of the literature this relation is assumed to hold also for fluctuations. However, in the next section we will show that this is an incorrect assumption. Therefore, results derived from this assumption and conclusions drawn are in principle suspect. (This misstep is understandable because most authors have not acquired as much insight into the nature of fluctuations phenomena as now.) Our investigation testifies to the necessity of a complete reexamination of all cases afresh. In fact, an evaluation of the noise kernel near the horizon seems unavoidable for the consideration of fluctuations and back-reaction issues.

[8] This means that possible effects on the Hawking radiation due to the fluctuations of the potential barrier for the radial mode functions will be missed by our analysis.

The solutions of this equation are given by

$$\langle \eta^2(M) \rangle_\xi = \langle \eta^2(M_0) \rangle_\xi \left(\frac{M_0}{M} \right)^4 + \frac{\epsilon_0}{4B} \left[\left(\frac{M_0}{M} \right)^4 - 1 \right]. \tag{16}$$

Provided that the fluctuations at the initial time corresponding to $M = M_0$ are negligible (much smaller than $\sqrt{\epsilon_0/4B} \sim 1$), the fluctuations become comparable to the background solution when $M \sim M_0^{2/3}$. Note that fluctuations of the horizon radius of order one in Planckian units do not correspond to Planck scale physics because near the horizon $\Delta R = r - 2M$ corresponds to a physical distance $L \sim \sqrt{M \Delta R}$, as can be seen from the line element for Schwarzschild, $ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$, by considering pairs of points at constant t. So $\Delta R \sim 1$ corresponds to $L \sim \sqrt{M}$, whereas a physical distance of order one is associated with $\Delta R \sim 1/M$, which corresponds to an area change of order one for spheres with those radii. One can, therefore, have initial fluctuations of the horizon radius of order one for physical distances well above the Planck length provided that we consider a black hole with a mass much larger than the Planck mass. One expects that the fluctuations for states that are regular on the horizon correspond to physical distances not much larger than the Planck length, so that the horizon radius fluctuations would be much smaller than one for sufficiently large black hole masses. Nevertheless, that may not be the case when dealing with states which are singular on the horizon, with estimated fluctuations of order $M^{1/3}$ or even $\sqrt{M}$ [9, 10, 11]. Confirming that the fluctuations are indeed so small for regular states and verifying how generic, natural and stable they are as compared to singular ones is a topic that we plan to address in future investigations.

Our result for the growth of the fluctuations of the size of the black hole horizon agrees with the result obtained by Bekenstein in Ref. [24] and implies that, for a sufficiently massive black hole (with a few solar masses or a supermassive black hole), the fluctuations become important before the Planckian regime is reached. Strictly speaking, one cannot expect that a linear treatment of the perturbations provides an accurate result when the fluctuations become comparable to the mean value, but it signals a significant growth of the fluctuations (at least until the nonlinear effects on the perturbation dynamics become relevant).

This growth of the fluctuations which was found by Bekenstein and confirmed here via the Einstein-Langevin equation seems to be in conflict with the estimate given by Wu and Ford in Ref. [12]. According to their estimate, the accumulated mass fluctuations over a period of the order of the black hole evaporation time ($\Delta t \sim M_0^3$) would be of the order of the Planck mass. The discrepancy is due to the fact that the first term on the right-hand side of Eq. (12), which corresponds to the perturbed expectation value $\langle \hat{T}_{ab}^{(1)}[g+h] \rangle_{\text{ren}}$ in Eq. (9), was not taken into account in Ref. [12]. The larger growth obtained here is a consequence of the secular effect of that term, which builds up in time (slowly at first, during most of the evaporation time, and becoming more significant at late times when the mass has changed substantially) and reflects the unstable nature of the background solution for an evaporating black hole.[9]

All this can be qualitatively understood as follows. Consider an evaporating black hole with initial mass M_0 and suppose that the initial mass is perturbed by an amount $\delta M_0 = 1$. The mean evolution for the perturbed black hole (without taking into account any fluctuations) leads to a mass perturbation that grows like $\delta M = (M_0/M)^2 \delta M_0 = (M_0/M)^2$, so that it becomes comparable to the unperturbed mass M when $M \sim M_0^{2/3}$, which coincides with the result obtained above. Such a coincidence has a simple explanation: the fluctuations of the Hawking flux slowly accumulated during most of the evaporating time, which are of the order of the Planck mass, as found by Wu and Ford, give a dispersion of that order for the mass distribution at the time when the instability of the small perturbations around the background solution start to become significant.

IV. CORRELATION BETWEEN OUTGOING AND INGOING ENERGY FLUXES

In this section we address the issue whether the simple relation between the energy flux crossing the horizon and the flux far from it also holds for the fluctuations. As we show in Appendix A, those correlations vanish for conformal fields in two-dimensional spacetimes. (The correlation function for the outgoing and ingoing null energy fluxes in an effectively two-dimensional model was explicitly computed in Refs. [31, 32] and it was indeed found to vanish.) On the

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[9] A clarification between our results and the claims by Hu, Raval and Sinha in Ref. [13] is in place here: both use the stochastic gravity framework and perform an analysis based on the Einstein-Langevin equation, so there should be no discrepancy. However, the claim in Ref. [13] was based on a qualitative argument that focused on the stochastic source, but missed the fact that the perturbations around the mean are unstable for an evaporating black hole. Once this is taken into account, agreement with the result obtained here is recovered.

other hand, in four dimensions the correlation function does not vanish in general and correlations between outgoing and ingoing fluxes do exist near the horizon (at least partially). This point is also explained in Appendix A.

For black hole masses much larger than the Planck mass, one can use the adiabatic approximation for the background mean evolution. Therefore, to lowest order in $L_{\rm H}$ one can compute the fluctuations of the stress tensor in Schwarzschild spacetime. In Schwarzschild, the amplitude of the fluctuations of $r^2 \hat{T}_v^r$ far from the horizon is of order $1/M^2$ ($= M^2/M^4$) when smearing over a correlation time of order M, which one can estimate for a hot thermal plasma in flat space [50, 51] (see also Ref. [12] for a computation of the fluctuations of $r^2 \hat{T}_v^r$ far from the horizon). The amplitude of the fluctuations of $r^2 \hat{T}_v^r$ is thus of the same order as its expectation value. However, their derivatives with respect to v are rather different: since the characteristic variation times for the expectation value and the fluctuations are M^3 and M respectively, $\partial(r^2 T_v^r)/\partial v$ is of order $1/M^5$ whereas $\partial(r^2 \xi_v^r)/\partial v$ is of order $1/M^3$. This implies an additional contribution of order $L_{\rm H}$ due to the second term in Eq. (6) if one radially integrates the same equation applied to stress tensor fluctuations (the stochastic source in the Einstein-Langevin equation). Hence, in contrast to the case of the mean value, the contribution from the second term in Eq. (6) cannot be neglected when radially integrating since it is of the same order as the contributions from the first term, and one can no longer obtain a simple relation between the outgoing energy flux far from the horizon and the energy flux crossing the horizon.

So far we have argued that the method employed for the mean value cannot be employed for the fluctuations. Although one expects that $r^2 \xi_v^r$ on the horizon and far from it will not be equal when including the contributions that results from radially integrating the second term in Eq. (6), one might wonder whether there is a possibility that those contributions would somehow cancel out. That possibility can, however, be excluded using the following argument. The smeared correlation function

$$\int dv h(v) \int dv' h(v') \, r^4 \langle \xi_v^r(v, r) \xi_v^r(v', r) \rangle_\xi, \tag{17}$$

where $h(v)$ is some appropriate smearing function and $\xi_v^r(v, r)$ has already been integrated over the whole solid angle, is divergent on the horizon but finite far from it. Therefore, $r^2 \xi_v^r$ on the horizon and far from it cannot be equal for each value of v.

Let us discuss in some more detail the fact that certain smearings of the quantity $r^4 \langle \xi_v^r(v, r) \xi_v^r(v', r) \rangle_\xi$ are divergent on the horizon but finite far from it. The smeared correlation function is related to the noise kernel as follows:

$$\int dv dv' h(v) h(v') \, r^4 \langle \xi_v^r(v, r) \xi_v^r(v', r) \rangle_\xi = r^4 \int dv dv' h(v) h(v') \int d\Omega d\Omega' N_{v\ v}^{r\ r}(v, r, \theta, \varphi; v', r, \theta', \varphi'). \tag{18}$$

The noise kernel is divergent in the coincident limit or for null-separated points. Smearing the noise kernel along all directions gives a finite result. However, although certain partial smearings also give a finite result, others do not. For instance, smearing along a timelike direction yields a finite result, whereas smearing on a spacelike hypersurface yields in general a divergent result [52]. On the other hand, the result of smearing along two "transverse" null directions (two null directions sharing the same orthogonal spacelike 2-surfaces) is also finite, but not for a smearing along just one null direction even if we also smear along the orthogonal spacelike directions. For $r > 2M$ Eq. (18) corresponds to a smearing along a timelike direction and gives a finite result for the smeared correlation function, but on the horizon it corresponds to a smearing along a single null direction and it is divergent.

A proof of the results described in the previous paragraph is provided in Appendices B and C by considering a product of smearing functions involving all directions and then taking different kinds of limits in which the smearing size along certain directions vanishes. One limit corresponds to taking a vanishing size for the smearing along one of the null directions and we refer to it as *smearing along null geodesics*. The other corresponds to taking vanishing sizes for certain spatial directions and we refer to it as *smearing along timelike curves*. The proof proceeds in two steps. First, in Appendix B it is shown for the flat space case. Then it is generalized to curved spacetimes in Appendix C using a quasilocal expansion in terms of Riemann normal coordinates.

V. DISCUSSION

Following the stochastic gravity program, in Sec. III we found that the spherically-symmetric fluctuations of the horizon size of an evaporating black hole become important at late times, and even comparable to its mean value when $M \sim M_0^{2/3}$, where M_0 is the mass of the black hole at some initial time when the fluctuations of the horizon radius

are much smaller than the Planck length.[10] This is consistent with the result previously obtained by Bekenstein in Ref. [24].

It is important to realize that for a sufficiently massive black hole, the fluctuations become significant well before the Planckian regime is reached. More specifically, for a solar mass black hole they become comparable to the mean value when the black hole radius is of the order of 10nm, whereas for a supermassive black hole with $M \sim 10^7 M_\odot$, that happens when the radius reaches a size of the order of 1mm. One expects that in those circumstances the low-energy effective field theory approach of stochastic gravity should provide a reliable description.

It is worth mentioning that other properties of the black hole can exhibit substantial fluctuations when taking into account the back reaction of Hawking radiation. As pointed out by Page in Ref. [53], by momentum conservation the fluctuations of the total momentum of the Hawking radiation emitted will cause a recoil of the position of the black hole, which will also fluctuate. According to Page's estimate, the spread of the distribution for the black hole position will become comparable to the size of the horizon by the time an energy of order $M^{1/3}$ has been emitted. This kind of fluctuations, which were not obtained in our calculation because we restricted our attention to spherically-symmetric metric perturbations, exhibit certain features that differ significantly from those of our result. The fluctuations that we found take a much longer time to build up and depend crucially on the unstable behavior of small perturbations of the semiclassical solution, with characteristic time-scales of the order of the evaporation time. On the contrary, this unstable behavior plays no major role in the growth of the position fluctuations. Furthermore, since this growth is still much slower than the emission rate of Hawking quanta, in the frame where the black hole is at rest the properties of the Hawking radiation being emitted remain essentially unchanged when the position fluctuations are taken into account, whereas the fluctuations of the horizon size do imply fluctuations in the temperature of the Hawking radiation.

Due to the nonlinear nature of the back-reaction equations, such as Eq. (10), the fact that the fluctuations of the horizon size can grow and become comparable to the mean value implies non-negligible corrections to the dynamics of the mean value itself. This can be seen by expanding Eq. (10) (evaluated on the horizon) in powers of η and taking the expectation value. Through order η^2 we get

$$
\begin{aligned}
\frac{d(M(v) + \langle \eta(v) \rangle_\xi)}{dv} &= -\left\langle \frac{B}{(M(v) + \eta(v))^2} \right\rangle_\xi \\
&= -\frac{B}{M^2(v)} \left[1 - \frac{2}{M(v)} \langle \eta(v) \rangle_\xi + \frac{3}{M^2(v)} \langle \eta^2(v) \rangle_\xi + O\left(\frac{\eta^3}{M^3}\right) \right].
\end{aligned}
\tag{19}
$$

When the fluctuations become comparable to the mass itself, the third term (and higher order terms) on the right-hand side is no longer negligible and we get non-trivial corrections to Eq. (8) for the dynamics of the mean value. These corrections can be interpreted as higher order radiative corrections to semiclassical gravity that include the effects of metric fluctuations on the evolution of the mean value. For instance, the third term on the right-hand side of Eq. (19) would correspond to a two-loop Feynman diagram involving a matter loop with an internal propagator for the metric perturbations (restricted to the spherically-symmetric sector in our case), as compared to just one matter loop, which is all that semiclassical gravity can account for.

Does the existence of the significant deviations for the mean evolution mentioned above imply that the results based on semiclassical gravity obtained by Bardeen and Massar in Refs. [2, 3] are invalid? Several remarks are in order. First of all, those deviations start to become significant only after a period of the order of the evaporation time when the mass of the black hole has decreased substantially. Secondly, since fluctuations were not considered in those references, a direct comparison cannot be established. However, we can compare the average of the fluctuating ensemble with their results. Doing so exhibits an evolution that deviates significantly when the fluctuations become important. Nevertheless, if one considers a single member of the ensemble at that time, its evolution will be accurately described by the corresponding semiclassical gravity solution until the fluctuations around that particular solution become important again, after a period of the order of the evaporation time associated with the new initial value of the mass at that time.

An interesting aspect that we have not addressed in this work, but which is worth investigating, is the quantum coherence of those fluctuations. It seems likely that, given the long time periods involved and the size of the fluctuations, the entanglement between the Hawking radiation emitted and the black hole spacetime geometry will effectively decohere the large horizon fluctuations, rendering them equivalent to an incoherent statistical ensemble.

In this paper we have taken a first step to put the study of metric fluctuations in black hole spacetimes on a firmer basis by considering a detailed derivation of the results from an appropriate formalism rather than using heuristic

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[10] Remember that for large black hole masses this can still correspond to physical distances much larger than the Planck length, as explained in Sec. III.

arguments or simple estimates. The spirit is somewhat analogous to the study of the mean back-reaction effect of Hawking radiation on a black hole spacetime geometry (both for black holes in equilibrium and for evaporating ones) by considering the solutions of semiclassical gravity in that case rather than just relying on simple energy conservation arguments. In order to obtain an explicit result from the stochastic gravity approach and compare with earlier work, in Sec. III we employed a simplifying assumption implicitly made in most prior work: the existence of a simple connection between the outgoing energy flux fluctuations far from the horizon and the negative energy flux fluctuations crossing the horizon. In Sec. IV we analyzed this assumption carefully and showed it to be invalid. This strongly suggests that one needs to study the stress tensor fluctuations from an explicit calculation of the noise kernel near the horizon. This quantity is obtainable from the stochastic gravity program and calculation is underway [54, 55, 56].

A possible way to compute the noise kernel near the horizon could be to use an approximation scheme based on a quasilocal expansion such as Page's approximation [38] or similar methods corresponding to higher order WKB expansions [41].[11] With these techniques one can obtain an approximate expression for the Wightman function of the matter fields, which is the essential object needed to compute the noise kernel. Unfortunately these approximations are only accurate for pairs of points with a small separation scale and break down when it becomes comparable to the black hole radius. Therefore, especial care is needed when studying the $l = 0$ multipole since that corresponds to averaging the noise kernel over the whole solid angle, which involves typical separations for pairs of points on the horizon of the order of the black hole radius, and one needs to make sure that the integral is dominated by the contribution from small angular separations. Alternatively, one might hope to gain some insight on the fluctuations near a black hole horizon by studying the fluctuations of the event horizon surrounding any geodesic observer in de Sitter spacetime, which exhibits a number of similarities with the event horizon of a black hole in equilibrium [57]. In contrast to the black hole case, it may be possible to obtain exact analytical results for de Sitter space due to its high degree of symmetry.

Furthermore, as explained in Sec. IV and shown in detail in Appendices B and C, the noise kernel smeared over the horizon is divergent, and so are the induced metric fluctuations. Hence, one cannot study the fluctuations of the horizon as a three-dimensional hypersurface for each realization of the stochastic source because the amplitude of the fluctuations is infinite, even when restricting one's attention to the $l = 0$ sector. Instead one should regard the horizon as possessing a finite effective width due to quantum fluctuations. In order to characterize its width one must find a sensible way of probing the metric fluctuations near the horizon and extracting physically meaningful information, such as their effect on the Hawking radiation emitted by the black hole. One possibility is to study how metric fluctuations affect the propagation of a bundle of null geodesics [31, 32, 49, 58, 59, 60]. One expects that this should provide a description of the effects on the propagation of a test field whenever the geometrical optics approximation is valid. However, preliminary analysis of simpler cases with a quantum field theory treatment suggest that when including quantum vacuum metric fluctuations the geometrical optics approximation becomes invalid. Another possibility, which seems to constitute a better probe of the metric fluctuations, is to analyze their effect on the two-point quantum correlation functions of a test field. The two-point functions characterize the response of a particle detector for that field and can be used to obtain the expectation value and the fluctuations of the stress tensor of the test field.

Finally, since the large fluctuations suggested in Refs. [7, 8, 9, 10] involve time-scales much shorter than the evaporation time (contrary to those considered in this paper) and high multipoles, one expects that for a sufficiently massive black hole the spacetime near the horizon can be approximated by Rindler spacetime (identifying the black hole horizon and the Rindler horizon) provided that we restrict ourselves to sufficiently small angular scales. Thus, analyzing the effect of including the interaction with the metric fluctuations on the two-point functions of a test field propagating in flat space, which is technically much simpler, could provide useful information for the black hole case.

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[11] Note, however, that in most of these approaches the state of the quantum fields is the Hartle-Hawking vacuum. For an evaporating black hole, one should consider the Unruh vacuum.

APPENDIX A: CORRELATIONS BETWEEN OUTGOING AND INGOING FLUXES IN $1+1$ AND $3+1$ DIMENSIONS

Any spacetime metric in $1+1$ dimensions is conformally flat (at least locally) and can be written as

$$ds^2 = -C(u,v)dudv. \tag{A1}$$

In terms of these null coordinates, the conservation equation for the stress tensor, $\nabla^a T_{ab} = 0$, reduces to

$$\partial_v \hat{T}_{uu} + \partial_u \hat{T}_{vu} - \Gamma^u_{uu} T_{vu} = 0, \tag{A2}$$
$$\partial_u \hat{T}_{vv} + \partial_v \hat{T}_{uv} - \Gamma^v_{vv} T_{uv} = 0, \tag{A3}$$

since all the other relevant Christoffel symbols vanish. Taking into account that $\Gamma^u_{uu} = \partial_u(\ln C)$, $\Gamma^v_{vv} = \partial_v(\ln C)$ and combining Eqs. (A2)-(A3) we get

$$\partial_v \hat{T}_{uu} + \partial_u \hat{T}_{vv} = -C\,\partial_t(\hat{T}_{uv}/C), \tag{A4}$$
$$\partial_v \hat{T}_{uu} - \partial_u \hat{T}_{vv} = C\,\partial_x(\hat{T}_{uv}/C), \tag{A5}$$

where we introduced the coordinates $t = (u+v)/2$ and $x = (v-u)/2$.

This result can be applied to the Schwarzschild geometry in $1+1$ dimensions identifying t with the usual Killing time and x with the Regge-Wheeler coordinate r_*. In that case we have $C(t,r) = (1 - 2M/r)$. If we consider a massless conformally coupled field in $1+1$ dimensions (conformal and minimal coupling are equivalent in that case), the trace of the stress tensor (which is related to the T_{uv} component) vanishes at the classical level and is entirely given by the trace anomaly $\langle \hat{T}^\mu_\mu \rangle_{\rm ren} = R/24\pi = M/6\pi r^3$. Since both the trace anomaly and the conformal factor are time-independent, the term on the right-hand side of Eqs. (A4) vanishes. Therefore, it follows from Eq. (A4) that the generation of left-moving and right-moving null mean fluxes is perfectly anticorrelated, which implies that the positive energy flux of outgoing Hawking radiation equals, in absolute value, the negative energy flux crossing the horizon. Moreover, from Eq. (A4) and the fact that $\langle \hat{T}_{uv} \rangle_{\rm ren}/C = -\langle \hat{T}^\mu_\mu \rangle_{\rm ren}/4 = -M/24\pi r^3$, which implies $C\,\partial_{r_*}(\langle \hat{T}_{uv} \rangle_{\rm ren}/C) = C^2\,M/6\pi r^4$, it is clear that the amount of anticorrelated mean fluxes generated tends to zero for large radii and is largest at $r = 3M$.

For energy flux fluctuations, the situation is very different. The trace anomaly does not fluctuate, *i.e.*, the trace of the stress tensor does not fluctuate for conformal fields [16]. Hence, in $1+1$ dimensions neither correlated nor anticorrelated fluctuations of the left-moving and right-moving null fluxes can be generated in the absence of other interactions: both fluxes are separately conserved.

On the other hand, although in $3+1$ dimensions one can try to use a similar argument when considering sectors with certain symmetries, the final conclusion is different. For instance, if one uses in Minkowski spacetime a coordinate system $\{u, v, y, z\}$ where $u = t - x$ and $v = t + x$ are null coordinates, the components $\hat{T}_{uy}$, $\hat{T}_{uz}$, $\hat{T}_{vy}$, $\hat{T}_{vz}$ and $\hat{T}_{yz}$ vanish in the sector which is rotationally invariant on the yz plane. In that sector, the u and v components of the conservation equation coincide with those in the 1+1 case, given by Eqs. (A2)-(A3), with vanishing Christoffel symbols. Therefore, the conditions for the generation of correlated or anticorrelated null fluxes are still given by Eqs. (A4)-(A5). However, in $3+1$ dimensions the $\hat{T}_{yy}$ and $\hat{T}_{zz}$ components also contribute to the trace of the stress tensor. Hence, although the trace does not fluctuate for conformal fields, $\hat{T}_{uv}$ does fluctuate and its fluctuations coincide with those of $(\hat{T}_{yy} + \hat{T}_{zz})/4$. Correlated and anticorrelated null energy flux fluctuations can thus be generated.

The discussion in the previous paragraph can be extended to a general spherically symmetric spacetime in the region where the gradient of the radial coordinate is spacelike. This can be done by considering angular coordinates for every sphere of constant radius and the coordinates associated with the two radial null directions orthogonal to them. Taking rotational symmetry into account, as done in the flat space case, one can provide an argument similar to that in $1+1$ dimensions. However, as in the flat space case, despite the absence of fluctuations in the trace of the stress tensor, $\hat{T}_{uv}$ will still fluctuate due to the fluctuations of $(\hat{T}_{\theta\theta} + \hat{T}_{\varphi\varphi}/\sin^2\theta)/4r^2$. Moreover, in this case there will be additional contributions due to scattering off the potential barrier (an effect that will be small near the horizon).

APPENDIX B: SMEARING OF THE NOISE KERNEL IN FLAT SPACE

In this appendix we will consider several kinds of smearings of the noise kernel for the Minkowski vacuum. In Sec. B 1 we study a product of smearing functions involving two null directions and the two orthogonal spatial directions, and analyze the limit in which the smearing size along one of the null directions vanishes, which is shown to be divergent. On the other hand, in Sec. B 2 a smearing along a timelike direction and the three orthogonal directions is considered

and it is shown that a finite smearing size along the timelike direction is sufficient to have a finite result. In contrast, in the limit of a vanishing smearing size along the timelike direction the result is always divergent, even for non-vanishing smearing sizes along all the spatial directions.

The noise kernel should be treated as distribution in spacetime coordinates. It has a divergent coincidence limit and it involves subtle integration prescriptions. However, the Fourier transforms of this kind of distributions are much simpler to deal with. Therefore, it is very convenient to perform our calculations in Fourier space and we will do so throughout this appendix.

The noise kernel $N_{abcd}(x, x') = (1/2)\langle\{\hat{T}_{ab}(x), \hat{T}_{cd}(x')\}\rangle - \langle\hat{T}_{ab}(x)\rangle\langle\hat{T}_{cd}(x')\rangle$ for a massless conformally coupled scalar field in the Minkowski vacuum state has been obtained for instance in Ref. [61] and is given in standard inertial coordinates by

$$N_{\mu\nu\rho\sigma}(x, x') = \frac{2}{3}\left(3\mathcal{D}_{\mu(\rho}\mathcal{D}_{\sigma)\nu} - \mathcal{D}_{\mu\nu}\mathcal{D}_{\rho\sigma}\right)N(x - x'), \tag{B1}$$

where $\mathcal{D}_{\mu\nu} \equiv \eta_{\mu\nu}\Box - \partial_\mu\partial_\nu$ and

$$N(x - x') = \frac{1}{(1920\pi)}\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-x')}\theta(-p^2). \tag{B2}$$

1. Smearing around a null geodesic

In this subsection we consider the case in which one smears around a null geodesic. Using the null coordinates $v = t + x$ and $u = t - x$, we define the smeared version of the kernel $N(x - x')$ as

$$\mathcal{N} \equiv \int du\, dv\, d^2x\, g(u)h(v)f(\vec{x})\int du'dv'd^2x'g(u')h(v')f(\vec{x}')N(v - v', u - u', \vec{x} - \vec{x}'), \tag{B3}$$

where we integrated the kernel with some smearing functions both for the null coordinates u and v, and for the orthogonal spatial directions. If we choose Gaussian smearing functions

$$g(u) = (2\pi\sigma_u^2)^{-\frac{1}{2}}\exp(-u^2/2\sigma_u^2), \tag{B4}$$

$$h(v) = (2\pi\sigma_v^2)^{-\frac{1}{2}}\exp(-v^2/2\sigma_v^2), \tag{B5}$$

$$f(\vec{x}) = (2\pi\sigma_r^2)^{-1}\exp(-\vec{x}^2/2\sigma_r^2), \tag{B6}$$

Eq. (B3) can be written as

$$\mathcal{N} = \frac{1}{(2\pi)^4\sigma_v^2\sigma_u^2\sigma_r^4}\int dVdUd^2X\, e^{-\frac{V^2}{\sigma_v^2}}e^{-\frac{U^2}{\sigma_u^2}}e^{-\frac{\vec{X}^2}{\sigma_r^2}}\int d\Delta_v d\Delta_u d^2\Delta\, e^{-\frac{\Delta_v^2}{4\sigma_v^2}}e^{-\frac{\Delta_u^2}{4\sigma_u^2}}e^{-\frac{\vec{\Delta}^2}{4\sigma_r^2}}N(\Delta_v, \Delta_u, \vec{\Delta}), \tag{B7}$$

where we introduced the semisum and difference variables $U = (u+u')/2$, $V = (v+v')/2$, $\vec{X} = (\vec{x}+\vec{x}')/2$, $\Delta_u = u' - u$, $\Delta_v = v' - v$ and $\vec{\Delta} = \vec{x}' - \vec{x}$. The integrals for U, V and $\vec{X}$ can be readily performed and yield the following result:

$$\mathcal{N} = \frac{1}{(4\pi)^2\sigma_v\sigma_u\sigma_r^2}\int d\Delta_v d\Delta_u d^2\Delta\, e^{-\frac{\Delta_v^2}{4\sigma_v^2}}e^{-\frac{\Delta_u^2}{4\sigma_u^2}}e^{-\frac{\vec{\Delta}^2}{4\sigma_r^2}}N(\Delta_v, \Delta_u, \vec{\Delta}). \tag{B8}$$

On the other hand, in order to compute the remaining integrals it is convenient to work in Fourier space. When considering the null coordinates v and u, it is useful to introduce the momenta $p_v = (p_x - p_t)/2$ and $p_u = -(p_t + p_x)/2$ so that Eq. (B2) becomes

$$N(x - x') = \frac{1}{(1920\pi)}\int \frac{d^4p}{(2\pi)^4}\, e^{i[p_v(v-v')+p_u(u-u')+\vec{p}\cdot(\vec{x}-\vec{x}')]}\theta(p_vp_u - \vec{p}^2), \tag{B9}$$

where we used the vector notation for the transverse components associated with the coordinates y and z. Eq. (B8) can then be expressed in Fourier space as

$$\mathcal{N} = \frac{1}{(1920\pi)}\frac{1}{(2\pi)^4}\int dp_v dp_u d^2p\, e^{-p_v^2\sigma_v^2}e^{-p_u^2\sigma_u^2}e^{-\vec{p}^2\sigma_r^2}\,\theta(p_vp_u - \vec{p}^2). \tag{B10}$$

One can then infer that $\mathcal{N}$ diverges as $\sigma_u \to 0$. This can be seen as follows. The integral in Eq. (B10) gets two identical contributions from the quadrants $(p_u, p_v > 0)$ and $(p_u, p_v < 0)$, whereas the remaining two quadrants give a vanishing contribution. Moreover, since the integrand is positive, the integral is also positive and greater than the same integral restricted to a smaller domain of integration. Taking all that into account, we have

$$
\begin{aligned}
\mathcal{N} &\geq \frac{2}{(1920\pi)(2\pi)^4} \int_{\substack{p_v p_u \geq \sigma_r^{-2} \\ p_v, p_u > 0}} dp_u dp_v e^{-p_u^2 \sigma_u^2} e^{-p_v^2 \sigma_v^2} \int_0^{\sigma_r^{-2}} d|\vec{p}|^2 e^{-1} \\
&\geq \frac{2e^{-1}}{\sigma_r^2} \frac{1}{(1920\pi)(2\pi)^4} \int_{2\sigma_v/\sigma_r^2}^{\infty} dp_u e^{-p_u^2 \sigma_u^2} \int_{\sigma_v^{-1}/2}^{\sigma_v^{-1}} dp_v e^{-1} \\
&= \frac{e^{-2}}{\sigma_r^2 \sigma_v} \frac{1}{(1920\pi)(2\pi)^4} \int_{2\sigma_v/\sigma_r^2}^{\infty} dp_u e^{-p_u^2 \sigma_u^2} \sim \frac{1}{\sigma_u \sigma_v \sigma_r^2}.
\end{aligned}
\tag{B11}
$$

The last integral is divergent if one takes $\sigma_u \to 0$ (at least for $\sigma_r \neq 0$ [12]). Thus, $\mathcal{N}$ diverges unless $\sigma_u \neq 0$.

Using the previous result, it is easy to discuss whether a smeared version of the actual noise kernel (including the differential operators), given by Eq. (B1), also diverges when $\sigma_u \to 0$. Each derivative in Eq. (B1) gives rise to an additional factor involving the momentum associated with the corresponding component. Additional factors involving powers of p_u, p_v, p_y^2 and p_z^2 (odd powers of p_y or p_z give a vanishing contribution) leave the argument employed in the previous paragraph unchanged and the same conclusions obtained when $\sigma_u \to 0$ hold for the smeared noise kernel as well (including the differential operators). For example we have $\mathcal{N}_{vvvv} \sim 1/\sigma_u \sigma_v^5 \sigma_r^2$ for the smeared version of the $vvvv$ component of the noise kernel.

On the other hand, when both $\sigma_v \neq 0$ and $\sigma_u \neq 0$ the smeared kernel $\mathcal{N}$ is finite even for $\sigma_r = 0$. This can be seen by taking $\sigma_r = 0$ in Eq. (B10). We then have

$$
\begin{aligned}
\mathcal{N} &= \frac{1}{(1920\pi)} \frac{1}{(2\pi)^4} \int dp_v dp_u d^2 p \, e^{-p_v^2 \sigma_v^2} e^{-p_u^2 \sigma_u^2} \theta(p_v p_u - \vec{p}^2) \\
&= \frac{1}{(1920\pi)} \frac{1}{(2\pi)^4} \int dp_v dp_u \pi p_v p_u e^{-p_v^2 \sigma_v^2} e^{-p_u^2 \sigma_u^2} \theta(p_v p_u) \\
&= \frac{2}{(1920\pi)} \frac{1}{(2\pi)^4} \int_0^{\infty} dp_v \int_0^{\infty} dp_u \pi p_v p_u e^{-p_v^2 \sigma_v^2} e^{-p_u^2 \sigma_u^2} \sim \frac{1}{\sigma_u^2 \sigma_v^2},
\end{aligned}
\tag{B12}
$$

which is finite for $\sigma_u, \sigma_v \neq 0$. It is also clear that the same conclusion applies to the smeared version of the noise kernel. For instance we have $\mathcal{N}_{vvvv} \sim 1/\sigma_u^2 \sigma_v^6$.

2. Smearing along a timelike curve and on a spacelike hypersurface

In this subsection we consider the smeared kernel $\mathcal{N}$ obtained when working with standard cartesian coordinates in Minkowski spacetime:

$$
\mathcal{N} \equiv \int dt \, d^3 x \, g(t) f(\vec{x}) \int dt' d^3 x' g(t') f(\vec{x}') N(t - t', \vec{x} - \vec{x}'),
\tag{B13}
$$

where we used the vector notation for the three spatial components x, y and z, and the Gaussian smearing functions are now

$$
g(t) = (2\pi \sigma_t^2)^{-\frac{1}{2}} \exp(-t^2/2\sigma_t^2),
\tag{B14}
$$

$$
f(\vec{x}) = (2\pi \sigma_r^2)^{-\frac{3}{2}} \exp(-\vec{x}^2/2\sigma_r^2).
\tag{B15}
$$

Note that σ_r corresponds now to the smearing size of the three spatial directions. Introducing the semisum and difference variables $T = (t + t')/2$ and $\Delta_t = t' - t$, Eq. (B13) can be rewritten as

$$
\mathcal{N} = \frac{1}{(2\pi)^4 \sigma_t^2 \sigma_r^6} \int dT d^3 X e^{-\frac{T^2}{\sigma_t^2}} e^{-\frac{\vec{X}^2}{\sigma_r^2}} \int d\Delta_t d^3 \Delta \, e^{-\frac{\Delta_t^2}{4\sigma_t^2}} e^{-\frac{\vec{\Delta}^2}{4\sigma_r^2}} N(\Delta_t, \vec{\Delta}).
\tag{B16}
$$

---

[12] The divergence of $\mathcal{N}$ as $\sigma_u \to 0$ can also be proven for $\sigma_r = 0$ (i.e., in the absence of smearing along the transverse spatial directions) by taking $\sigma_r = 0$ in Eq. (B10) and replacing σ_r^2 with an arbitrary but fixed positive value in Eq. (B11).

After integrating over T and $\vec{X}$ we have

$$\mathcal{N} = \frac{1}{(4\pi)^2 \sigma_t \sigma_r^3} \int d\Delta_t d^3\Delta \, e^{-\frac{\Delta_t^2}{4\sigma_t^2}} e^{-\frac{\vec{\Delta}^2}{4\sigma_r^2}} N(\Delta_t, \vec{\Delta}), \tag{B17}$$

which can be equivalently rewritten in Fourier space as

$$\mathcal{N} = \frac{1}{(1920\pi)} \frac{1}{(2\pi)^4} \int dp_t d^3 p \, e^{-p_t^2 \sigma_t^2} e^{-\vec{p}^2 \sigma_r^2} \, \theta(p_t^2 - \vec{p}^2). \tag{B18}$$

A similar argument to that employed in the previous subsection can be used to show that in this case $\mathcal{N}$ diverges as $\sigma_t \to 0$. The integral in Eq. (B18) gets two identical contributions from the intervals $p_t < 0$ and $p_t > 0$. We will also take into account that since the integrand is positive, the integral is also positive and greater than the same integral restricted to a smaller domain of integration, so that we have

$$\begin{aligned}
\mathcal{N} &\geq \frac{2}{(1920\pi)(2\pi)^4} \int_{\sigma_r^{-1}}^{\infty} dp_t e^{-p_t^2 \sigma_t^2} \int_0^{\sigma_r^{-1}} 4\pi |\vec{p}|^2 d|\vec{p}| e^{-1} \\
&= \frac{8\pi e^{-1}}{3\sigma_r^3} \frac{1}{(1920\pi)(2\pi)^4} \int_{\sigma_r^{-1}}^{\infty} dp_t e^{-p_t^2 \sigma_t^2} \sim \frac{1}{\sigma_t \sigma_r^3}.
\end{aligned} \tag{B19}$$

The last integral is divergent if one takes $\sigma_t \to 0$ (at least for $\sigma_r \neq 0$ [13]). Thus, $\mathcal{N}$ diverges unless $\sigma_t \neq 0$.

Using the previous result, it is easy to discuss whether a smeared version of the actual noise kernel, given by Eq. (B1), also diverges when $\sigma_t \to 0$. Each derivative in Eq. (B1) gives rise to an additional factor involving the momentum associated with the corresponding component. Additional factors involving powers of p_t^2, p_x^2, p_y^2 and p_z^2 (odd powers of p_t, p_x, p_y or p_z give a vanishing contribution) leave the argument employed in the previous paragraph unchanged and the same conclusions obtained when $\sigma_t \to 0$ hold for the smeared noise kernel as well. For example one has $\mathcal{N}_{tttt} \sim 1/\sigma_t \sigma_r^7$. This result is in agreement with that obtained in Ref. [52]

On the other hand, when $\sigma_t \neq 0$ the smeared noise kernel is finite even for $\sigma_r = 0$. This can be seen by taking $\sigma_r = 0$ in Eq. (B18). We then have

$$\mathcal{N} = \frac{1}{(1920\pi)} \frac{1}{(2\pi)^4} \int dp_t d^3 p \, e^{-p_t^2 \sigma_t^2} \theta(p_t^2 - \vec{p}^2) = \frac{1}{(1920\pi)} \frac{1}{(2\pi)^4} \int dp_t \frac{4\pi}{3} p_t^3 e^{-p_t^2 \sigma_t^2} \sim \frac{1}{\sigma_t^4}, \tag{B20}$$

which is finite for $\sigma_t \neq 0$. It is also clear that the same conclusion applies to the smeared version of the noise kernel, with $\mathcal{N}_{tttt} \sim 1/\sigma_t^8$.

APPENDIX C: GENERALIZATION TO CURVED SPACE, ARBITRARY HADAMARD GAUSSIAN STATES AND GENERAL SMEARING FUNCTIONS

The results obtained for the Minkowski vacuum in flat space can be generalized to curved space and arbitrary Gaussian Hadamard states. They also apply to more general smearing functions. This will be shown in this appendix.

The key ingredient is the fact that the Wightman function for any Hadamard state has the following form in a sufficiently small normal neighborhood of an arbitrary spacetime:[14]

$$G^+(x, x') = \frac{u(x, x')}{\sigma_+(x, x')} + v(x, x') \ln \sigma_+(x, x') + w(x, x'), \tag{C1}$$

where $\sigma_+(x, x')$ is the geodetic interval (one half of the geodesic distance) for the geodesic connecting the pair of points x and x' with an additional small imaginary component added to the timelike coordinates (this prescription

---

[13] The divergence of $\mathcal{N}$ as $\sigma_t \to 0$ can also be proven for $\sigma_r = 0$ (*i.e.*, in the absence of smearing along the spatial directions) by taking $\sigma_r = 0$ in Eq. (B18) and replacing σ_r^2 with an arbitrary but fixed positive value in Eq. (B19).

[14] For a general spacetime it may not be guaranteed that the series has a non-vanishing radius of convergence rather than being an asymptotic series [62]. However, for an analytic spacetime it can be proven that the radius of convergence is non-zero for globally Hadamard states [62, 63]. We will restrict ourselves to analytic spacetimes in this appendix, which is anyway the case for the spacetimes considered in the rest of the paper.

will be defined more precisely below); u, v and w are smooth functions with v and w expandable as

$$v(x, x') = \sum_{n=0}^{\infty} v_n(x, x')\sigma^n(x, x'), \tag{C2}$$

$$w(x, x') = \sum_{n=0}^{\infty} w_n(x, x')\sigma^n(x, x'). \tag{C3}$$

where $u(x, x')$, $v_n(x, x')$ and $w_n(x, x')$ satisfy the Hadamard recursion relations, which uniquely determine $u(x, x')$ and $v(x, x')$. On the other hand, $w_0(x, x')$ is not uniquely determined and contains the information on the particular state that one is considering, the remaining $w_n(x, x')$ are also determined once a particular choice of $w_0(x, x')$ has been made. Note that for the Minkowski vacuum in flat space $v(x, x')$ and $w(x, x')$ vanish and $u(x, x')$ is simply given by a constant. Other Hadamard states in flat space have a non-vanishing $w(x, x')$ while $u(x, x')$ and $v(x, x')$ remain unchanged.

The biscalar functions $u(x, x')$, $v_n(x, x')$ and $w_n(x, x')$ can in turn be expanded in the following way:

$$A(x, x') = \sum_{m=0}^{\infty} A_{a_1 \cdots a_m}(x)\, \sigma^{a_1} \cdots \sigma^{a_m}, \tag{C4}$$

where $\sigma^a \equiv g^{ab}\nabla_b\sigma$. Furthermore, it will be convenient to employ Riemann normal coordinates, which can always be introduced in a normal neighborhood. Given a set of normal coordinates $\{y^\mu\}$, the geodetic interval can be simply expressed as $\sigma(y, y') = (1/2)(y^\mu - y'^\mu)(y^\nu - y'^\nu)\eta_{\mu\nu}$. The prescription for σ_+ mentioned above corresponds then to $\sigma_+(y, y') = (1/2)[-(y^0 - y'^0 - i\varepsilon)^2 + (\vec{y} - \vec{y}')^2]$.

Given a smearing function $s(x)$, one can consider the following smearing of the product of two Wightman functions $N(x, x') = \mathrm{Re}[G^+(x, x')G^+(x, x')]$:

$$\mathcal{N} = \int d^4x\, \sqrt{-g(x)}\, s(x) \int d^4x'\, \sqrt{-g(x')}\, s(x')\, N(x, x'). \tag{C5}$$

Next, one changes from a set of absolute coordinates for each one of the the two points at which the kernel $N(x, x')$ is evaluated to a set of absolute coordinates for the first point and a set of relative coordinates for the location of the second point with respect to the first one. In particular we will choose Riemann normal coordinates for the relative location of the second point (in order to study the divergences in the coincidence limit it is sufficient, by taking small enough smearing sizes, to consider small enough convex neighborhoods where both normal coordinates can be defined and the series in Eq. (C1) is convergent). Eq. (C6) then becomes

$$\mathcal{N} = \int d^4x\, \sqrt{-g(x)}\, s(x) \int d^4y\, \sqrt{-\tilde{g}_x(y)}\, \tilde{s}_x(y)\, \tilde{N}(x, y). \tag{C6}$$

For simplicity, in our discussion below we will consider smearing functions $\tilde{s}_x(y)$ which are Gaussian and independent of x. However, in Sec. C 3 we will explain how our results can be extended to more general smearing functions. We also note that we will not analyze the integrals in x since those should be finite (we are considering globally Hadamard states and regular smearing functions): only the integrals in y are relevant for the UV divergences associated with the coincidence limit, which corresponds to $y \to 0$. There could still be IR divergences arising from the integrals in x, but we will be considering smearing functions which decay sufficiently fast so that this is not the case.

Taking all the previous considerations into account, it becomes clear that when calculating the smeared kernel $\mathcal{N}$, the most divergent terms from $N(x, x') = \mathrm{Re}[G^+(x, x')G^+(x, x')]$ will be of the form $1/\sigma_+^2(x, x')$, $\ln \sigma_+(x, x')/\sigma_+(x, x')$ or $1/\sigma_+(x, x')$. When expressed in normal coordinates, the contribution to $\mathcal{N}$ due to a term of the first kind, which will be denoted by $\mathcal{N}_1$ and corresponds to the product of two $1/\sigma_+(x, x')$ terms, coincides with the expression for the Minkowski vacuum in flat space. Hence, one can directly apply the results obtained in Appendix A. Furthermore, one expects that the leading divergence to the smeared function $\mathcal{N}$ in the various limits of vanishing smearing sizes will come entirely from that kind of terms and that the other terms will only give rise to subleading divergences. If this is true, the main conclusions in Appendix A will also apply to general Hadamard Gaussian states in curved spacetime. Let us, therefore, study more carefully the contributions to $\mathcal{N}$ from the different kinds of divergent terms and check that this is indeed the case (other possible divergent terms in addition to the three kinds mentioned above correspond to multiplying one of those three by some positive power of σ^a, and they will be discussed in Sec. C 1 b).

In order to analyze the contribution from the other two kinds of divergent terms, denoted by $\mathcal{N}_2$ and $\mathcal{N}_3$, we will proceed analogously to Appendix A and work in Fourier space for the relative normal coordinates. We start by considering a term of the form $\ln \sigma_+(x, x')/\sigma_+(x, x')$. Using the same Gaussian smearing functions as in Appendix A

(as explained above, we do not include any dependence on the absolute coordinate for the first point), its contribution to $\mathcal{N}$ when smearing around a null geodesic is given by

$$\mathcal{N}_2 = \frac{1}{(2\pi)^4} \int dp_v dp_u d^2p \, e^{-p_v^2 \sigma_v^2} e^{-p_u^2 \sigma_u^2} e^{-\vec{p}^2 \sigma_r^2} \frac{1}{2} \left[L(p) + (L(-p))^* \right], \tag{C7}$$

where $L(p_v, p_u, \vec{p}^2)$ is the Fourier transform of $\ln(\sigma_+(y,y')/\lambda^2)/\sigma_+(y,y')$ and we need the second term inside the square brackets because we are interested in the real part of the product of two Wightman functions. An explicit expression for $L(p_v, p_u, \vec{p}^2)$ can be determined using the following two Fourier transforms:

$$\frac{1}{\sigma_+(x,x')} = \frac{1}{\pi} \int d^4p \, e^{ip(y-y')} \, \theta(-p^0) \delta(-p^2), \tag{C8}$$

$$\ln(\sigma_+(y,y')/\lambda^2) = -2 \int d^4p \, e^{ip(y-y')} \lim_{m \to 0} \left\{ -\frac{1}{\pi} \theta(-p^0) \frac{d}{dm^2} \delta(-p^2 - m^2) \right.$$
$$\left. + \left[\frac{1}{2} \ln(2m^2 \lambda^2) + \gamma - 1 \right] \delta^4(p) \right\}. \tag{C9}$$

where Eq. (C9) was derived in Appendix A.2 of Ref. [64]. We can then write $L(p_v, p_u, \vec{p}^2)$ as a convolution of those Fourier transforms:

$$L(p) = -4(2\pi)^3 \int d^4q \, \theta(-p^0 + q^0)) \delta(-(p-q)^2)$$
$$\times \lim_{m \to 0} \left\{ -\frac{1}{\pi} \theta(-q^0) \frac{d}{dm^2} \delta(-q^2 - m^2) + \left[\frac{1}{2} \ln(2m^2 \lambda^2) + \gamma - 1 \right] \delta^4(q) \right\}. \tag{C10}$$

After a certain amount of calculation, one obtains the following result for $L(p)$:

$$L(p) = 2(2\pi)^3 \left\{ \theta(-p^0) \, \mathcal{P}f \left[\theta(-p^2) \frac{1}{p^2} \right] - \left[\ln(2\lambda^2) + \gamma - 1 \right] \theta(-p^0) \delta^4(-p^2) \right\}, \tag{C11}$$

where $\mathcal{P}f$ denotes Hadmard's finite part prescription (a generalization of the principal value prescription) whose precise definition can be found in Refs. [64, 65, 66]. We will use this result in the next subsections to compute in Fourier space the contribution $\mathcal{N}_2$ to the smeared kernel $\mathcal{N}$ for different kinds of smearings and analyze under what conditions it is finite.

On the other hand, from Eq. (C8) one can immediately see that the contribution from the term $1/\sigma_+(x,x')$ is simply given by

$$\mathcal{N}_3 = \frac{1}{2\pi} \int dp_v dp_u d^2p \, e^{-p_v^2 \sigma_v^2} e^{-p_u^2 \sigma_u^2} e^{-\vec{p}^2 \sigma_r^2} \delta(p_u p_v - \vec{p}^2). \tag{C12}$$

1. Smearing around null geodesics

a. Contributions from $\mathcal{N}_2$ and $\mathcal{N}_3$

As we found in Appendix A, when considering a smearing of the noise kernel for the Minkowski vacuum around a null geodesic, $\mathcal{N}$ diverges as $1/\sigma_u \sigma_v \sigma_r^2$ in the limit of small σ_u (as long as $\sigma_r \neq 0$, otherwise it diverges as $1/\sigma_u^2 \sigma_v^2$). Whereas the contribution $\mathcal{N}_1$ from terms of the form $1/\sigma_+^2(x,x')$ will exhibit the same divergent behavior, we will show in this subsection that $\mathcal{N}_2$ and $\mathcal{N}_3$, the other contributions to $\mathcal{N}$, are arbitrarily smaller than $\mathcal{N}_1$ for sufficiently small σ_u or σ_v.

Let us start with $\mathcal{N}_3$. First, one rewrites Eq. (C12) as follows:

$$\mathcal{N}_3 = \frac{1}{2\pi} \int dp_v dp_u d^2p \, e^{-p_v^2 \sigma_v^2} e^{-p_u^2 \sigma_u^2} e^{-\vec{p}^2 \sigma_r^2} \delta(p_u p_v - \vec{p}^2)$$
$$= \int_0^\infty dp_v \int_0^\infty dp_u e^{-p_v^2 \sigma_v^2} e^{-p_u^2 \sigma_u^2} \int_0^\infty d|\vec{p}| \, e^{-|\vec{p}|^2 \sigma_r^2} \delta(\sqrt{p_u p_v} - |\vec{p}|)$$
$$= \int_0^\infty dp_v \int_0^\infty dp_u e^{-p_v^2 \sigma_v^2} e^{-p_u^2 \sigma_u^2} e^{-p_u p_v \sigma_r^2}$$
$$= \int_0^\infty d\xi e^{-\xi \sigma_r^2} \int_0^\infty \frac{dp_v}{p_v} e^{-p_v^2 \sigma_v^2} e^{-\frac{\xi^2}{p_v^2} \sigma_u^2}, \tag{C13}$$

where we introduced the new variable $\xi = p_v p_u$ in the last equality. Next, using the positivity of the integrand, the fact that the original integral was invariant under interchange of p_u and p_v, and the fact that value of the exponentials is always equal or less than one, one can derive the following bound for small σ_u and σ_v:

$$\mathcal{N}_3 \leq -\frac{1}{2\sigma_r^2}(\ln \sigma_u + \ln \sigma_v) + O(1), \tag{C14}$$

where the higher-order terms involve positive powers of σ_u and σ_v (when σ_v is not small, one has an expansion only in terms of σ_u and the $\ln \sigma_v$ term is absent). On the other hand, for $\sigma_r = 0$ the bound is $\mathcal{N}_3 \leq (\pi^2/8)(1/\sigma_u\sigma_v)$.

Let us now turn our attention to $\mathcal{N}_2$. Substituting Eq. (C11) into Eq. (C7), we get

$$\mathcal{N}_2 = -\int dp_v dp_u d^2p \, e^{-p_v^2 \sigma_v^2} e^{-p_u^2 \sigma_u^2} e^{-\vec{p}^2 \sigma_r^2}\left\{ \mathcal{P}f\left[\theta(p_u p_v - \vec{p}^2)\frac{1}{p_u p_v - \vec{p}^2}\right] + \left[\ln(2\lambda^2) + \gamma - 1\right]\delta^4(p_u p_v - \vec{p}^2)\right\}. \tag{C15}$$

The contribution from the second term inside the curly brackets has the same form as $\mathcal{N}_3$. Hence, we need to concentrate on the first term. After a lengthy calculation one can derive the following bound:

$$\mathcal{N}_2 \leq L\frac{1}{\sqrt{2\sigma_u \sigma_v}\,\sigma_r} + O(\ln \sigma_u, \ln \sigma_v), \tag{C16}$$

where L is a constant of order 1. On the other hand, for $\sigma_r = 0$ the bound is $\mathcal{N}_2 \leq L/\sigma_u\sigma_v$. We can see that $\mathcal{N}_2$ dominates over $\mathcal{N}_3$ in the limit of small σ_u or σ_v. Nevertheless, it is still $\mathcal{N}_1$ that provides the leading contribution in that limit.

b.   Remaining terms

The contribution from the remaining terms in $N(x, x')$ to the smeared function $\mathcal{N}$, which can be seen from terms of the different kinds that we have already analyzed multiplying them by some positive powers of σ and σ^a can be analyzed as follows. If one is working with Riemann normal coordinates and the corresponding Fourier variables, each σ factor will give rise to a momentum d'Alembertian $(1/2)\eta_{\mu\nu}(\partial/\partial p_\mu)(\partial/\partial p_\nu)$ acting on the Fourier transform of that term in $N(x, x')$. Similarly, each σ^μ factor will give rise to a linear differential operator $(\partial/\partial p_\mu)$ acting on the Fourier transform. Integrating by parts in the Fourier space expression for $\mathcal{N}$ so that the d'Alembertian acts on the Fourier transform of the smearing functions, it will produce a factor of the form $(-\sigma_u^2 \sigma_v^2 p_u p_v + 2\sigma_r^4 \vec{p}^2 - 2\sigma_r^2)$. Proceeding analogously with the linear operator $(\partial/\partial p_\mu)$, one gets the factors $2\sigma_u^2 p_u$, $2\sigma_v^2 p_v$ and $2\sigma_r^2 p_j$ (where the index j corresponds to one of the two orthogonal spatial components) when μ equals u, v or j respectively.

Next, one needs to see how the results of the integrals in Sec. C 1 a change when the integrand is multiplied by positive powers of p_u, p_v and p_j. Odd powers of p_j for any j give a vanishing result since the integrals and integrands there are symmetric under a sign change of p_j, whereas every $(p_j)^2$ factor can be written as $(-1/2)(\partial/\partial\sigma_r^2)$. Similarly, every factor p_u^2 or p_v^2 can be written as $(-1/2)(\partial/\partial\sigma_u^2)$ or $(-1/2)(\partial/\partial\sigma_v^2)$ respectively.[15] On the other hand, having an odd power of p_u only or p_v only gives a vanishing result because the integrals and integrands in Sec. C 1 a are symmetric under a simultaneous sign change of p_u and p_v. However, odd powers of $p_u p_v$ do not vanish in general because the integrands are not symmetric under interchange of p_u or p_v only. Therefore, one needs to check how the main results for the integrals in Sec. C 1 a change when the integrands are multiplied by a $p_u p_v$ factor. In Eq. (C13) it gives rise to a factor ξ, which implies an additional $1/\sigma_r^2$ factor multiplying the final result for $\mathcal{N}_3$ in Eq. (C14). One can similarly find that an additional $p_u p_v$ factor in Eq. (C15) also implies a $1/\sigma_r^2$ factor multiplying the final result for $\mathcal{N}_2$ in Eq. (C16).

We are finally in a position to discuss the effects of σ and σ_a factors multiplying the contributions to the smeared kernel $\mathcal{N}$. Each power of p_u^2 and p_v^2 (including their σ_u^4 and σ_v^4 accompanying factors) will typically give rise to σ_u^2 and σ_v^2 factors respectively. A $p_u p_v$ factor (with its accompanying $\sigma_u^2 \sigma_v^2$ factor) will give rise to a $\sigma_u^2 \sigma_v^2/\sigma_r^2$ factor. And each p_j^2 factor (with its accompanying σ_r^4 factor) will give rise to a σ_r^2 factor. Thus, we see that the divergent behavior in the limit of small σ_u remains unchanged or even gets improved. (An analogous conclusion would apply in the limit of small σ_r.) It then follows that the behavior of $\mathcal{N}$ in the small σ_u limit is still dominated by the flat space vacuum contribution $\mathcal{N}_1$.

---

[15] In general when two functions satisfy a certain inequality that does not imply that their derivatives will satisfy it. This will, however, be the case when considering the bounds derived in the previous sections in order to analyze the leading divergent behavior in the limit $\sigma_u \to 0$.

The actual noise kernel involves a number of functions multiplying the kernel $N(x, x') = \mathrm{Re}[G^+(x, x')G^+(x, x')]$ and differential operators acting on it. When using relative Riemann normal coordinates for the location of the second point with respect to the first one, the part of these linear operators that depends on the relative coordinates of the second point can be entirely expressed in terms of functions and tensor fields (such as the metric and the curvature tensors) as well as partial derivatives. That is not the case in general for the operators associated with the first spacetime point, but the dependence on the first point does not exhibit a divergent UV behavior.

The functions multiplying $N(x, x')$ can be expanded in terms of the relative Riemann normal coordinates, which involves powers of σ and σ^a that can be treated as explained in the previous subsubsection. As we saw, they either leave the divergent behavior in the limit of small σ_u unchanged or decrease the degree of divergence.

On the other hand, the partial derivatives $(\partial/\partial y^\mu)$ simply correspond to ip_μ factors in Fourier space, which can also be dealt with as discussed in the previous subsubsection. Even powers of p_u increase the degree of divergence in the limit of small σ_u: every p_u^2 factor gives rise to a $1/\sigma_u^2$ factor. All others leave the degree of divergence unchanged. Moreover, since the p_u^2 affect both the kind of terms contributing to $\mathcal{N}_1$ and those contributing to $\mathcal{N}_2$ and $\mathcal{N}_3$, the conclusion that the leading divergent behavior when $\sigma_u \to 0$ is given by the Minkowski vacuum result remains unchanged for the actual noise kernel.

2. Smearing along timelike geodesics and on spacelike hypersurfaces

The results for smearings of the noise kernel on spacelike hypersurfaces and along timelike geodesics obtained for the vacuum state in Minkowski can be generalized proceeding analogously to what was done in the previous subsection for smearings along null geodesics.

Let us start by considering $\mathcal{N}_3$:

$$
\begin{aligned}
\mathcal{N}_3 &= \frac{1}{2\pi} \int dp_t d^3p \, e^{-p_t^2 \sigma_t^2} e^{-\vec{p}^2 \sigma_r^2} \delta(p_t^2 - \vec{p}^2) \\
&= 4 \int_0^\infty dp_t e^{-p_t^2 \sigma_t^2} \int_0^\infty d|\vec{p}| \, |\vec{p}| e^{-|\vec{p}|^2 \sigma_r^2} \delta(p_t - |\vec{p}|) \\
&= 4 \int_0^\infty dp_t \, p_t e^{-p_t^2 \sigma_t^2} e^{-p_t^2 \sigma_r^2}.
\end{aligned}
\tag{C17}
$$

Hence, we have

$$
\mathcal{N}_3 = \frac{2}{\sigma_t^2 + \sigma_r^2},
\tag{C18}
$$

which is finite provided that $\sigma_t \neq 0$ or $\sigma_r \neq 0$.

Let us now turn our attention to $\mathcal{N}_2$. Applying the result in Eq. (C11) to this case, we get

$$
\mathcal{N}_2 = - \int dp_t d^3p \, e^{-p_t^2 \sigma_t^2} e^{-\vec{p}^2 \sigma_r^2} \left\{ \mathcal{P}f \left[\theta(p_t^2 - \vec{p}^2) \frac{1}{p_t^2 - \vec{p}^2} \right] + \left[\ln(2\lambda^2) + \gamma - 1 \right] \delta^4(p_t^2 - \vec{p}^2) \right\}.
\tag{C19}
$$

The contribution from the second term inside the curly brackets has the same form as $\mathcal{N}_3$. Hence, we only need to concentrate on the first term. After a slightly lengthy calculation, one gets the following bound for $\mathcal{N}_2$ (when $\sigma_r \neq 0$):

$$
\mathcal{N}_2 < \frac{C_1}{\sigma_r^2} + \frac{C_2}{\sigma_t^2},
\tag{C20}
$$

where C_1 and C_2 are positive dimensionless constants which are finite provided that $\sigma_t \neq 0$ (they behave like $-\ln \sigma_t$ in the limit of small σ_t). On the other hand, for $\sigma_r = 0$ one has a bound given by

$$
|\mathcal{N}_2| < \frac{D}{\sigma_t^2},
\tag{C21}
$$

where D is some positive dimensionless constant which is finite provided that $\sigma_t \neq 0$ (it behaves like $-\ln \sigma_t$ in the limit of small σ_t). This shows that just a temporal smearing is enough to render $\mathcal{N}_2$ finite. Of course $\mathcal{N}_2$ will also be finite if, in addition to the temporal smearing, some but not all of the (orthogonal) spatial directions have a non-vanishing smearing size.

Proceeding similarly to Secs. C 1 b-C 1 c, one can argue that factors involving positive powers of σ and σ^a as well as the derivative operators in configuration space do not alter the main conclusion in this subsection. Thus, a smearing along the timelike direction is enough to have a finite result for the smeared versions of $N(x, x')$ and the actual noise kernel.

3. More general smearing functions

In Secs. C 1 and C 2 we considered Gaussian smearing functions for the relative normal coordinates. However, as pointed out at the beginning of this appendix, when transforming from a pair of sets of *absolute* coordinates for the two points where the noise kernel is evaluated to one set of *absolute* coordinates and one set of *relative* ones, the form of the smearing functions will change in general. That will also happen when transforming to Riemann normal coordinates if one had initially chosen a different kind of relative coordinates. Furthermore, even in the flat space case one may be interested in considering other kinds of smearing functions. For instance, one may wish to consider a smearing function adapted to a spherical surface rather than a plane.

The results obtained in these appendices can be extended to more general smearing functions, making it possible to cover the situations described in the previous paragraph. The essential idea is simple: the previous results can be generalized for smearing functions which can be locally approximated by the Gaussian smearing functions in Riemann normal coordinates considered so far. More specifically, by "locally approximated" we mean that in those coordinates the new smearing functions can be expressed as the Gaussian smearing functions times a factor involving an expansion in positive powers of σ and σ^a. The procedure described in Sec. C 1 b can then be employed to show that the main results remain unchanged. Moreover, the detailed form of the smearing functions for large values of the Riemann normal coordinates (or even outside the range where they can be defined) is not relevant when concerned with the divergent behavior (when certain smearing sizes tend to zero) as we are are here. Or put in a different way, even though in general the form of the smearing function can be significantly distorted when transforming from the original coordinates to relative Riemann normal coordinates, this effect becomes less and less important when considering sufficiently small smearing sizes (which is in any case the relevant regime to study the UV divergent behavior): for sufficiently small scales the coordinate transformation is characterized by the linear map between tangent spaces (the Jacobian matrix evaluated in the coincidence limit) plus small corrections involving positive powers of σ and σ^a, which can be dealt with following the procedure described in Sec. C 1 b.

These points can be illustrated with a simple example. Consider a spatial smearing function adapted to a sphere in flat space with the following form (in spherical coordinates):

$$f(r, \theta) = \frac{1}{(2\pi)^{3/2} \sigma_r r_0^2 \sigma_\theta^2 \, b(\sigma_\theta)} \exp\left[-\frac{(r - r_0)^2}{2\sigma_r^2}\right] \exp\left[-\frac{\theta^2}{2\sigma_\theta^2}\right], \tag{C22}$$

where $b(\sigma_\theta)$ is some dimensionless function which ensures the proper normalization of the smearing function and tends to 1 when $\sigma_\theta \to 0$ [note also that for θ we actually have half a Gaussian since its domain is $(0, \pi)$]. This can be written in terms of Riemann normal coordinates (cartesian coordinates in this case) adapted to the plane tangent to the sphere at the point $(r = r_0, \ \theta = 0)$ as follows:

$$
\begin{aligned}
f(x, y, z) \ = \ & \frac{1}{(2\pi)^{3/2} \sigma_r r_0^2 \sigma_\theta^2 \, b(\sigma_\theta)} \exp\left[-\frac{(z - r_0)^2}{2\sigma_r^2}\right] \exp\left[-\frac{x^2 + y^2}{2 r_0^2 \sigma_\theta^2}\right] \\
& \times \left[1 + O\left(\frac{(z - r_0)^3}{r_0^3 \sigma_\theta^2}, \frac{(x^2 + y^2)(z - r_0)}{r_0^3 \sigma_\theta^2}, \frac{(x^2 + y^2)(z - r_0)}{r_0 \sigma_r^2}\right)\right].
\end{aligned}
\tag{C23}
$$

We can see that for any given $r_0 > 0$, if one chooses sufficiently small σ_r and σ_θ, the higher-order terms become negligible (in the region where there is not a large suppression due to the exponential factors they are very small). Thus, $f(x, y, z)$ corresponds to a Gaussian smearing function in cartesian coordinates with $\sigma_x = \sigma_y = r_0 \sigma_\theta$ and $\sigma_z = \sigma_r$ plus small corrections.

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